Problems in oscillations and waves

1. *Imagine a straight shaft between Europe and Australia passing through the center of the Earth. If a body enters the shaft, it will be acted upon by a force that is directed towards the center of the Earth and is directly proportional to the distance from the center of the Earth. Calculate how long it would take a body that was dropped into the shaft to travel from Europe to Australia and back, and what speed the body would have when passing through the center of the Earth. Gravitational acceleration on the surface of the Earth is* $g = 9{,}81\,\mathrm{m\,s^{-2}}$ *a polomer Zeme je* $R_E = 6370\,\mathrm{km}$.

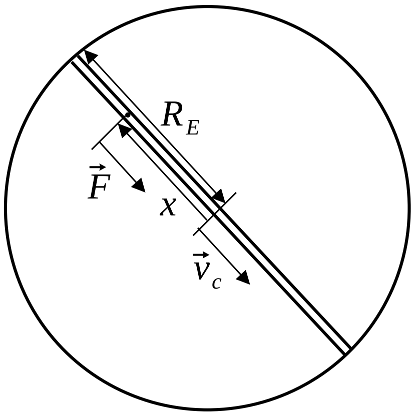

Fig. 1

The body in the shaft will be acted upon by a force whose magnitude is

$$F = -kx \ ,$$

from Newton's second law it follows

$$ma = F \ ,$$

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx \ .$$

On the surface of the Earth, the force is equal to the weight of the body

$$kR_E = mg \ ,$$

from which the constant is

$$k = m\frac{g}{R_E} \ ,$$

using which it is possible to write the equation of motion in the form

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{g}{R_E} x \ ,$$

which is the equation of harmonic motion

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 x \ ,$$

whose angular frequency is

$$\omega = \sqrt{\frac{g}{R_E}} \ ,$$

and its solution has the form

$$x = x_0 \cos(\omega t + \alpha) \ .$$

Since at time $t = 0\,\mathrm{s}$ s the position of the body was $x = R_E$, the following applies

$$x_0 = R_E \ ,$$

$$\alpha = 0 \ ,$$

so the equation describing the movement of the body will be

$$x = R_E \cos\left(\sqrt{\frac{g}{R_E}} t\right) \ .$$

The journey of a body from Europe to Australia and back is a period of motion

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{R_E}{g}} \ ,$$

after substituting numerical values

$$T = 2\pi \sqrt{\frac{6{,}37 \cdot 10^6\,\mathrm{m}}{9{,}81\,\mathrm{m\,s^{-2}}}} = 5063\,\mathrm{s} = 84\,\mathrm{min} \ .$$

The velocity of the body can be expressed as

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\left[R_E \cos\left(\sqrt{\frac{g}{R_E}} t\right)\right]}{\mathrm{d}t} = -\sqrt{R_E g} \sin\left(\sqrt{\frac{g}{R_E}} t\right) \ .$$

For a body in the center of the Earth, the following applies

$$R_E \cos\left(\sqrt{\frac{g}{R_E}} t_c\right) = 0 \ ,$$

hence the time is

$$t_c = \frac{\pi}{2}\sqrt{\frac{R_Z}{g}} \ .$$

After substituting into the velocity of the body, the velocity of the body at the center of the Earth will be

$$v_c = -\sqrt{R_Z g}\sin\left(\sqrt{\frac{g}{R_E}}t_c\right) = -\sqrt{R_E g}\sin\left(\sqrt{\frac{g}{R_E}}\,\frac{\pi}{2}\sqrt{\frac{R_E}{g}}\right) = -\sqrt{R_E g} \ ,$$

where the negative symbol means that the v_s direction is opposite to the x direction, after substituting the numerical values, the velocity of the body in the center of the Earth will be

$$v_c = \sqrt{6{,}37 \cdot 10^6\,\mathrm{m}\ .\ 9{,}81\,\mathrm{m\,s^-1}} = 7905\,\mathrm{m\,s^{-1}} \ .$$

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2. *Two bodies with massesi $m_1 = 5\,\mathrm{kg}$ and $m_2 = 3\,\mathrm{kg}$ are connected by a spring whose spring constant is $k = 100\,\mathrm{N\,m^{-1}}$. We bring the bodies closer to each other, thereby compressing the spring and then releasing the bodies. Calculate the period of oscillation of the bodies.*

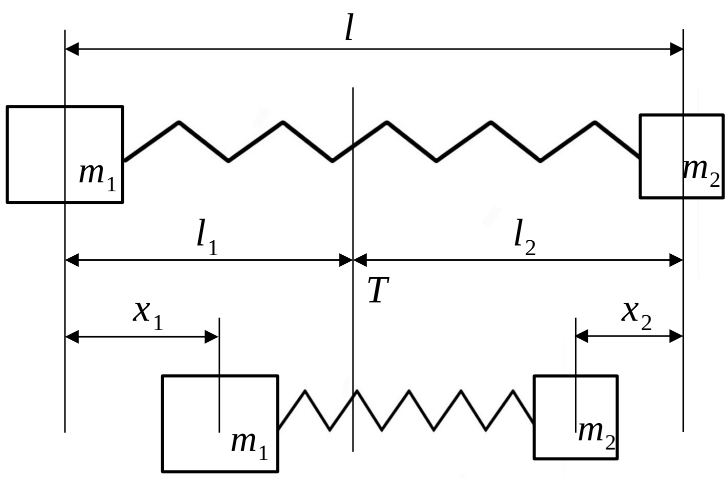

Fig. 2

If the origin of the coordinate system is in the common center of gravity of the bodies, the following will apply

$$0 = \frac{-m_1 l_1 + m_2 l_2}{m_1 + m_2} \ ,$$

where l_1 and l_2 are the distances of the centers of gravity of individual bodies from their common center of gravity. It follows that

$$m_1 l_1 = m_2 l_2 \ .$$

Since it is an isolated system, after compressing the spring, the position of the center of gravity will not change and will apply

$$0 = \frac{-m_1(l_1 - x_1) + m_2(l_2 - x_2)}{m_1 + m_2} \ ,$$

where x_1 and x_2 are the deviations of the bodies from their equilibrium positions. It follows that

$$m_1 x_1 = m_2 x_2 \ .$$

The force acting on the first body will be

$$F_1 = -k(x_1 + x_2) \ .$$

Because

$$x_2 = x_1 \frac{m_1}{m_2} \ ,$$

the force acting on the first body can be expressed as

$$F_1 = -k \frac{m_1 + m_2}{m_2} x_1 = -k_1 x_1 \ ,$$

where

$$k_1 = k \frac{m_1 + m_2}{m_2} \ .$$

The angular frequency of the first body will be

$$\omega_1 = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}}$$

and the period of the first body will be

$$T_1 = \frac{2\pi}{\omega_1} = 2\pi \sqrt{\frac{m_1 m_2}{k(m_1 + m_2)}} \ ,$$

after substitution

$$T_1 = 2\pi \sqrt{\frac{5\,\mathrm{kg} \ . \ 3\,\mathrm{kg}}{100\,\mathrm{N\,m^{-1}} \ . \ (5\,\mathrm{kg} + 3\,\mathrm{kg})}} = 0{,}86\,\mathrm{s} \ .$$

The force acting on the other body will be

$$F_2 = -k(x_1 + x_2) \ .$$

Because

$$x_1 = x_2 \frac{m_2}{m_1} \ ,$$

it is possible to express the force acting on the second body as

$$F_2 = -k \frac{m_1 + m_2}{m_1} x_2 = -k_2 x_2 \ ,$$

where

$$k_2 = k \frac{m_1 + m_2}{m_1} \ .$$

The angular frequency of the second body's motion will be

$$\omega_2 = \sqrt{\frac{k_2}{m_2}} = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}}$$

and the period of motion of the second body will be

$$T_2 = \frac{2\pi}{\omega_2} = 2\pi \sqrt{\frac{m_1 m_2}{k(m_1 + m_2)}} \ .$$

The period of movement of the first and second bodies are therefore the same $T_1 = T_2 = 0{,}86\,\mathrm{s}$.

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3. *Calculate the period of harmonic motion of a body of mass* $m = 100\,\mathrm{g}$ *suspended on a spring. A force* $F_1 = 0{,}2\,\mathrm{N}$ *is needed to extend the spring by* $x_1 = 10\,\mathrm{cm}$.

The equation of motion of a body with mass m performing harmonic motion has the form

$$m \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -kx \ ,$$

where x is the displacement of the body from the equilibrium position and k is the spring constant. The equation has a solution in the form

$$x = x_0 \cos(\omega t + \alpha) \ ,$$

where x_0 is the amplitude and α is the phase constant of the motion. For the angular frequency applies

$$\omega = \sqrt{\frac{k}{m}} \, .$$

A force F_1 is required to extend the spring by x_1, therefore

$$F_1 = kx_1 \, ,$$

for the spring constant it follows

$$k = \frac{F_1}{x_1} \, .$$

The period of harmonic motion can be calculated as

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{mx_1}{F_1}} \, ,$$

after substitution

$$T = 2\pi \, . \, \sqrt{\frac{0{,}1\,\mathrm{kg} \, . \, 0{,}1\,\mathrm{m}}{0{,}2\,\mathrm{N}}} = 1{,}4\,\mathrm{s} \, .$$

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4. *Mechanical work* $A_1 = 0{,}25\,\mathrm{J}$ *is required to extend the spring by* $x_1 = 5\,\mathrm{cm}$*. What will be the angular frequency of a body with mass* $m = 0{,}5\,\mathrm{kg}$*, that will oscillate on this spring?*

The force required to extend the spring by x is

$$F = kx \, .$$

The mechanical work when stretched by x_1 will therefore be

$$A_1 = \int_0^{x_1} F\mathrm{d}x = \int_0^{x_1} kx\mathrm{d}x = \left[\frac{1}{2}kx^2\right]_0^{x_1} = \frac{1}{2}kx_1^2 \, ,$$

from which it is possible to express the spring constant

$$k = \frac{2A_1}{x_1^2} \, .$$

The harmonic motion of a body is described by the equation of motion

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx \, .$$

Its solution has the form

$$x = x_0 \cos(\omega t + \alpha) \ ,$$

where x_0 is the amplitude and α is the phase constant of the motion. For the angular frequency applies

$$\omega = \sqrt{\frac{k}{m}} \ .$$

After substituting for spring constant, the angular frequency will be

$$\omega = \sqrt{\frac{2A_1}{mx_1^2}}$$

and after substituting numerical values

$$\omega = \sqrt{\frac{2\ .\ 0{,}25\,\mathrm{J}}{0{,}5\,\mathrm{kg}\ .\ (0{,}05\,\mathrm{m})^2}} = 20\,\mathrm{s}^{-1} \ .$$

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5. *A horizontal board performs a harmonic motion in the horizontal direction with a period of* $T = 3\,\mathrm{s}$. *The body lying on the board starts to slide when the amplitude of oscillations reaches the valueu* $x_0 = 0{,}5\,\mathrm{m}$. *What is the coefficient of friction between the body and the board?*

A frictional force acts on the body on the board, the magnitude of which is given by the multiplication of the coefficient of friction and the normal force

$$F_t = \mu N \ .$$

The magnitude of the normal force is equal to the product of the mass of the body and the acceleration of gravity

$$N = mg \ ,$$

therefore, the magnitude of the frictional force will be

$$F_t = \mu mg \ .$$

Because the body moves together with the board, it is in a non-inertial frame of reference, and in addition to the frictional force, the body also has an inertial force

$$F_z = -ma \ .$$

The motion of the board is described by the function

$$x = x_0 \cos(\omega t + \alpha) \ ,$$

from which for the speed of the plate follows

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}[x_0 \cos(\omega t + \alpha)]}{\mathrm{d}t} = -x_0 \omega \sin(\omega t + \alpha) \ ,$$

from which for the acceleration of the board follows

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}[-x_0 \omega \sin(\omega t + \alpha)]}{\mathrm{d}t} = -x_0 \omega^2 \cos(\omega t + \alpha) \ ,$$

which can be used to express the magnitude of the inertial force

$$F_z = m x_0 \omega^2 \cos(\omega t + \alpha) = F_{z0} \cos(\omega t + \alpha) \ ,$$

where the amplitude of the inertial force is

$$F_{z0} = m x_0 \omega^2 \ .$$

The body starts to slide when the amplitude of the inertial force equals the frictional force

$$F_{z0} = F_t \ ,$$

that is, when it will be valid

$$m x_0 \omega^2 = \mu m g \ ,$$

where the angular frequency can be expressed using the period

$$\omega = \frac{2\pi}{T} \ ,$$

from which the coefficient of friction follows

$$\mu = \frac{4\pi^2 x_0}{T^2 g} \ ,$$

after substitution

$$\mu = \frac{4\pi^2 \ . \ 0{,}5\,\mathrm{m}}{(3\,\mathrm{s})^2 \ . \ 9{,}81\,\mathrm{m\,s^{-2}}} = 0{,}22 \ .$$

6. *The particle performs damped harmonic motion, the dependence of the particle's position on time is given by the function* $x = 5\,\mathrm{cm}\, e^{-1{,}4\,\mathrm{s}^{-1}t} \cos(1{,}6\pi\ \mathrm{s}^{-1}t)$. *Calculate the damping coefficient, the logarithmic decrement of the damping, the time it takes for the amplitude of the oscillations to drop to one-hundredth of the original value, and the angular frequency at which the particle would oscillate if the damping force stopped acting.*

The position of a particle in damped harmonic motion is described by a function

$$x = x_0 e^{-bt} \cos(\omega t + \alpha)\ ,$$

from which it follows that the damping coefficient is

$$b = 1{,}4\,\mathrm{s}^{-1}\ ,$$

and the logarithmic decrement of the damping is

$$\delta = bT = b\frac{2\pi}{\omega} = 1{,}4\,\mathrm{s}^{-1}\frac{2\pi}{1{,}6\pi\ \mathrm{s}^{-1}} = 0{,}875\ .$$

Time for the amplitude to drop to one hundredth

$$x_0 e^{-bt_1} = \frac{x_0}{100}\ ,$$

will be

$$t_1 = \frac{\ln 100}{b} = \frac{\ln 100}{1{,}4\,\mathrm{s}^{-1}} = 3{,}29\,\mathrm{s}\ .$$

For the angular frequency of the damped harmonic oscillator applies

$$\omega = \sqrt{\omega_0^2 - b^2}\ ,$$

from which it is possible to express the angular frequency of motion without damping

$$\omega_0 = \sqrt{\omega^2 + b^2} = \sqrt{(1{,}6\pi\ \mathrm{s}^{-1})^2 + (1{,}4\,\mathrm{s}^{-1})^2} = 5{,}22\,\mathrm{s}^{-1}\ .$$

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7. *The result of adding two harmonic motions on a line is the motion described by the equation* $x = x_0 \cos\left(2\,\mathrm{s}^{-1}\ .\ t\right) \cos\left(50\,\mathrm{s}^{-1}\ .\ t\right)$. *Calculate the angular frequencies of the original harmonic motions and the angular frequency of the shocks of the resulting motion.*

When adding two harmonic motions

$$x = x_0 \cos(\omega_1 t) \ ,$$

$$x = x_0 \cos(\omega_2 t) \ ,$$

on the basis of the principle of superposition the resulting motion is

$$x = x_0 \cos(\omega_1 t) + x_0 \cos(\omega_2 t) \ .$$

Using the relationship

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \ ,$$

it is possible to write the resulting motion as

$$x = 2x_0 \cos\left(\frac{\omega_1-\omega_2}{2}t\right)\cos\left(\frac{\omega_1+\omega_2}{2}t\right) \ .$$

The angular frequencies of the original motions can be obtained by solving the system of equations

$$\frac{\omega_1-\omega_2}{2} = 2\,\mathrm{s}^{-1} \ ,$$

$$\frac{\omega_1+\omega_2}{2} = 50\,\mathrm{s}^{-1} \ ,$$

which implies

$$\omega_1 = 52\,\mathrm{s}^{-1} \ ,$$

$$\omega_2 = 48\,\mathrm{s}^{-1} \ .$$

The resulting motion can be written as a harmonic motion

$$x = A\cos\left(\frac{\omega_1+\omega_2}{2}t\right) \ ,$$

whose amplitude changes slowly

$$A = \left|2x_0 \cos\left(\frac{\omega_1-\omega_2}{2}t\right)\right| \ ,$$

the period of this amplitude is

$$T_A = \frac{2\pi}{\frac{|\omega_1-\omega_2|}{2}} = \frac{4\pi}{|\omega_1-\omega_2|} \ .$$

Because two amplifications and two attenuations occur in one period of the amplitude, that is, two shocks, for their period applies

$$T_s = \frac{T_A}{2} = \frac{2\pi}{|\omega_1-\omega_2|}$$

and the angular frequency of the shocks is

$$\omega_s = \frac{2\pi}{T_s} = |\omega_1-\omega_2| = \left|52\,\mathrm{s}^{-1} - 48\,\mathrm{s}^{-1}\right| = 4\,\mathrm{s}^{-1} \ .$$

8. *The wave travels through a medium, the dislacement of medium particles is described by the function* $u = A\cos 2\pi(bt - hx)$, *where* $A = 2\cdot 10^{-6}\,\mathrm{m}$, $b = 5000\,\mathrm{s}^{-1}$ *a* $h = 1\,\mathrm{m}^{-1}$. *Calculate the wavelength, frequency, period, amplitude, velocity of the wave, the maximum value of the velocity and acceleration of particle oscillations and write a function for the same wave traveling in the opposite direction.*

A function describing the wave in the direction of the x axis

$$u = u_0\cos(\omega t - kx)\ ,$$

where k is the wave number

$$k = \frac{2\pi}{\lambda}\ ,$$

can be rewritten to the form

$$u = u_0\cos 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)\ ,$$

which implies that the wavelength of the wave is

$$\lambda = \frac{1}{h} = \frac{1}{1\,\mathrm{m}^{-1}} = 1\,\mathrm{m}\ .$$

The frequency ot the wave is

$$f = b = 5000\,\mathrm{s}^{-1}\ ,$$

the period of the wave is

$$T = \frac{1}{f} = \frac{1}{b} = 2\cdot 10^{-4}\,\mathrm{s}\ ,$$

the amplitude of the wave is

$$u_0 = A = 2\cdot 10^{-6}\,\mathrm{m}\ ,$$

and the speed of the wave is

$$c = \frac{\lambda}{T} = \frac{b}{h} = \frac{1\,\mathrm{m}}{2\cdot 10^{-4}\,\mathrm{s}} = 5000\,\mathrm{m\,s}^{-1}\ .$$

The speed of the oscillation of the particles is

$$v = \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}\left[A\cos 2\pi(bt - hx)\right]}{\mathrm{d}t} = -2\pi bA\sin 2\pi(bt - hx)\ .$$

The maximum value of the speed of the oscillations of the particles is

$$v_{max.} = 2\pi bA = 2\pi\ .\ 5000\,\mathrm{s}^{-1}\ .\ 2\cdot 10^{-6}\,\mathrm{m} = 0{,}0628\,\mathrm{m\,s}^{-1}\ .$$

The acceleration of the oscillation of the particles is

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}\left[-2\pi bA\sin 2\pi(bt-hx)\right]}{\mathrm{d}t} = -(2\pi b)^2 A\cos 2\pi(bt-hx)\ .$$

The maximum value of the acceleration of the oscillations of the particles is

$$a_{max.} = (2\pi b)^2 A = (2\pi\ .\ 5000\,\mathrm{s}^{-1})^2\ .\ 2\cdot 10^{-6}\,\mathrm{m} = 1973{,}9\,\mathrm{m\,s}^{-2}\ .$$

For a wave traveling in the opposite direction, the following apply

$$x \to -x\ ,$$

therefore, the equation describing the wave traveling in the opposite direction will have the form

$$u = A\cos 2\pi(bt+hx)\ .$$

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9. *The speed of sound in steel can be determined by creating a wave in a steel rod fixed in the middle, which vibrates the air in a Kundt tube, in which a standing wave is created. Calculate the speed of sound in steel and the tensile modulus of steel if the distance between two standing wave nodes in air is* $x = 8\,\mathrm{cm}$*, the length of the rod is* $l = 1{,}2\,\mathrm{m}$*, the speed of sound in air is* $v = 340\,\mathrm{m\,s}^{-1}$ *and the density of steel is* $\rho = 7800\,\mathrm{kg\,m}^{-3}$.

The condition for the formation of standing waves in the air and the rod, as well as the formation of resonance, is

$$f = f'\ .$$

For the frequency of the wave in the air applies

$$f = \frac{v}{\lambda}\ ,$$

where v and λ are the speed and the wavelength of the wave in air. For the frequency of the wave in the rod applies

$$f' = \frac{v'}{\lambda'}\ ,$$

where v' and λ' are the speed and the wavelength of wave in the rod. Therefore, it is possible to rewrite the condition for the formation of standing waves in the form

$$\frac{v}{\lambda} = \frac{v'}{\lambda'} \,,$$

from which for the speed of wave in the rod it follows

$$v' = v \frac{\lambda'}{\lambda} \,,$$

where the wavelength in air can be determined as twice the distance between two nodes

$$\lambda = 2x$$

and the wavelength in the rod can be determined as twice the length of the rod

$$\lambda' = 2l \,,$$

using which for the speed of sound in steel follows

$$v' = v \frac{2l}{2x} = v \frac{l}{x} \,,$$

after substitution

$$v' = 340\,\mathrm{m\,s^{-1}} \,.\, \frac{1{,}2\,\mathrm{m}}{0{,}08\,\mathrm{m}} = 5100\,\mathrm{m\,s^{-1}} \,.$$

The wave equation has the general form

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} \,.$$

Using Hooke's law, it is possible to derive the equation for waves in steel

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2} \,,$$

therefore, the speed of sound in steel will be

$$v'^2 = \frac{E}{\rho} \,,$$

from which for the tensile modulus of steel follows

$$E = v'^2 \rho \,,$$

after substitution

$$E = (5100\,\mathrm{m\,s^{-1}})^2 \,.\, 7800\,\mathrm{kg\,m^{-3}} = 2{,}03 \cdot 10^{11}\,\mathrm{Pa} \,.$$

10. *The whistle with closed end produces a tone of fundamental frequency* $f = 130{,}5\,\mathrm{Hz}$. *Calculate the length of the whistle and the fundamental frequency, if the end of the whistle is open. The speed of sound in air* $v = 340\,\mathrm{m\,s^{-1}}$.

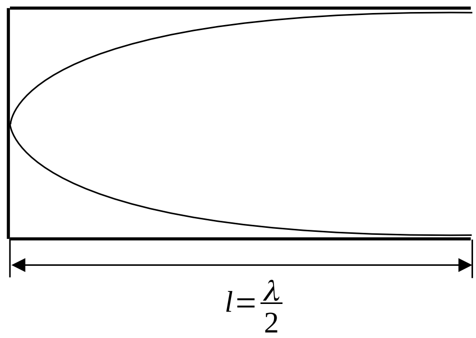

Fig. 3

If one end of the whistle is closed and the other is open (Fig. 43), there will be a antinode at closed end of the node at the open end. For the length of the whistle and the wavelength of the fundamental frequency follows

$$l = \frac{\lambda}{4}\,.$$

Because between the wavelength and the frequency apply

$$f = \frac{v}{\lambda}\,,$$

it is possible to express the length of the whistle as

$$l = \frac{v}{4f}\,,$$

after substitution

$$l = \frac{340\,\mathrm{m\,s^{-1}}}{4\,.\,130{,}5\,\mathrm{Hz}} = 0{,}65\,\mathrm{m}\,.$$

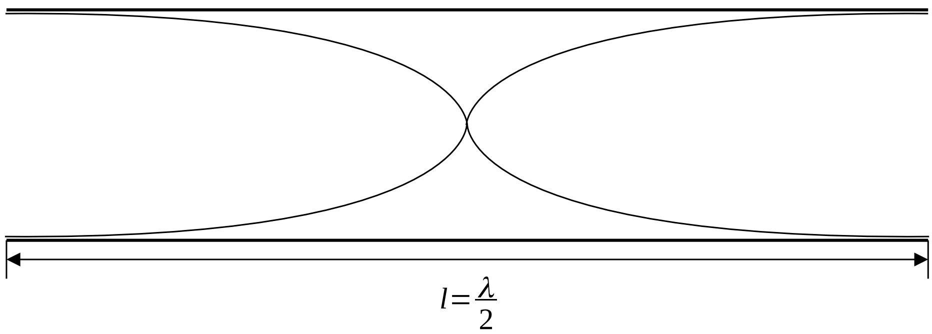

Fig. 4

If both ends of the whistle are open (Fig. 44), there will be the antinodes at both ends of the whistle. For the length of the whistle and the wavelength of the fundamental frequancy follows

$$l = \frac{\lambda}{2} \,.$$

Because between the wavelength and the frequency apply

$$f = \frac{v}{\lambda} \,,$$

it is possible to express the length of the whistle as

$$f = \frac{v}{2l} \,,$$

after substitution

$$f = \frac{340\,\mathrm{m\,s^{-1}}}{2\,.\,0{,}65\,\mathrm{m}} = 261{,}5\,\mathrm{Hz} \,.$$