# Problems in optics

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1. *Two monochromatic plane electromagnetic waves of the same frequency, polarized in the same plane, with amplitudes $E_{01} = 5\,\mathrm{V\,m^{-1}}$ and $E_{02} = 7\,\mathrm{V\,m^{-1}}$ propagate in the same direction in vacuum. Calculate the resulting wave intensity if the waves are a) incoherent b) coherent and the phase shift between them is $\delta = \frac{\pi}{3}$.*

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a) In the superposition of two incoherent waves, the resulting wave intensity is equal to the sum of the wave intensities

$$I = I_1 + I_2 \ ,$$

which implies

$$I = \frac{1}{2}c\epsilon_0 E_{01}^2 + \frac{1}{2}c\epsilon_0 E_{02}^2 = \frac{1}{2}c\epsilon_0 (E_{01}^2 + E_{02}^2) \ ,$$

after insertion

$$I = \frac{1}{2} \ . \ 3 \cdot 10^8\,\mathrm{m\,s^{-1}} \ . \ 8{,}854 \cdot 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}} \ . \ [(5\,\mathrm{V\,m^{-1}})^2 + (7\,\mathrm{V\,m^{-1}})^2] =$$

$$= 0{,}098\,\mathrm{W\,m^{-2}} \ .$$

b) In the superposition of two coherent waves, the resulting wave intensity is

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta \ ,$$

which implies

$$I = \frac{1}{2}c\epsilon_0 E_{01}^2 + \frac{1}{2}c\epsilon_0 E_{02}^2 + c\epsilon_0 E_{01}E_{02}\cos\delta = \frac{1}{2}c\epsilon_0 \left(E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos\delta\right) \ ,$$

after insertion

$$I = \frac{1}{2} \ . \ 3 \cdot 10^8\,\mathrm{m\,s^{-1}} \ . \ 8{,}854 \cdot 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}} \ . \ [(5\,\mathrm{V\,m^{-1}})^2 + (7\,\mathrm{V\,m^{-1}})^2 +$$

$$+2 \ . \ 5\,\mathrm{V\,m^{-1}} \ . \ 7\,\mathrm{V\,m^{-1}} \ . \ \cos\frac{\pi}{3}] = 0{,}145\,\mathrm{W\,m^{-2}} \ .$$

2. *A plano-convex lens is laid on a planar plate. Light of wavelength* $\lambda = 598\,\mathrm{nm}$ *is incident perpendicularly on the flat side of the lens. In the reflected light the Newton's rings are observed. The radius of the fifth dark ring is* $r_5 = 5\,\mathrm{mm}$. *Calculate the radius of the convex surface of the lens and the radius of the fourth dark ring.*

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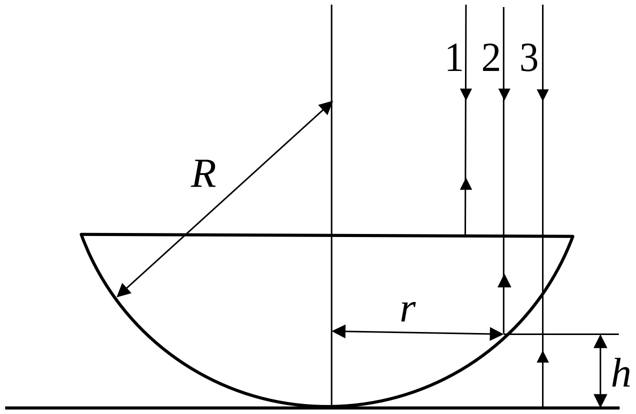


Fig. 1

Part of the light incident on the flat surface of the lens is reflected from this surface (ray 1) and part penetrates into the lens. In the penetrating part of the light, some light is reflected from the convex surface of the lens (ray 2) and some penetrates further and is reflected up to the plane plate (ray 3). Ray 1 does not interfere with either ray 2 or ray 3 because their path difference is greater than the coherent length of the light. Interference occurs between rays 2 and 3. Their path difference is

$$\Delta = 2h + \frac{\lambda}{2}\,,$$

because ray 3 has twice passed through a layer of air with refractive index $n = 1$ and on reflection from the optically denser medium on the bottom plate there is a phase change of $\pi$, which corresponds to a path difference of $\frac{\lambda}{2}$. The interference minimum condition is

$$2h + \frac{\lambda}{2} = (2m+1)\frac{\lambda}{2} \qquad m = 0, 1, 2, \dots\ .$$

The Pythagorean Theorem implies

$$(R-h)^2 + r^2 = R^2\,,$$

from which it is possible to express

$$r^2 = h(2R - h) \ .$$

Since $h << R$, the following holds

$$r^2 = h(2R - h) \approx h2R \ ,$$

which implies

$$h = \frac{r^2}{2R} \ .$$

When inserted to the condition for the interference minimum, then

$$r_m = \sqrt{mR\lambda} \ ,$$

from which it is possible to express the radius of the lens as

$$R = \frac{r_m^2}{m\lambda} \ ,$$

from the values for the fifth interference minimum

$$R = \frac{r_5^2}{5\lambda} \ .$$

the lens radius can be calculated

$$R = \frac{(5 \cdot 10^{-3}\,\mathrm{m})^2}{5 \ . \ 598 \cdot 10^{-9}\,\mathrm{m}} = 8{,}36\,\mathrm{m} \ .$$

The radius of the fourth interference minimum is

$$r_4 = \sqrt{4\lambda R} \ ,$$

after inserting

$$r_4 = \sqrt{4 \ . \ 598 \cdot 10^{-9}\,\mathrm{m} \ . \ 8{,}36\,\mathrm{m}} = 4{,}47\,\mathrm{mm} \ .$$

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3. *A plano-convex lens is laid on a planar plate. Light is incident perpendicularly on the flat side of the lens. In the reflected light we observe Newton's rings. If the space between the lens and the plate is filled with liquid, the radius of the fourth dark ring will be the same as the radius of the third dark ring when there was air in the space between the lens and the plate. Calculate the index of refraction of the liquid.*

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Since the space between the lens and the planar plate is filled by a liquid with refractive index $n$, the difference in the optical paths of the rays reflected from the convex part of the lens and the planar plate is

$$\Delta = 2nh + \frac{\lambda}{2} \, .$$

The condition of the interference minimum is

$$2nh + \frac{\lambda}{2} = (2m+1)\frac{\lambda}{2} \qquad m = 0, 1, 2, \dots \, ,$$

because

$$h = \frac{r^2}{2R} \, ,$$

the radius of the m-th dark ring is

$$r_m = \sqrt{\frac{mR\lambda}{n}} \, ,$$

so the radius of the fourth dark ring, if there is a liquid in the space, is

$$r_4 = \sqrt{\frac{mR\lambda}{n}} \, .$$

and the radius of the third dark ring, if there is air in the space, is

$$r_3 = \sqrt{mR\lambda} \, ,$$

because for air $n = 1$. The condition of equality of radii

$$r_4 = r_3 \, ,$$

implies

$$\sqrt{\frac{mR\lambda}{n}} = \sqrt{mR\lambda} \, ,$$

from which the refractive index of the liquid can be expressed

$$n = \frac{4}{3} \approx 1{,}33 \, .$$

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4. *To prevent light loss by reflection, the glass plate, whose refractive index is $n_1 = 1{,}66$, is covered on both sides with a thin covering of transparent material. What must be the index of refraction of the cover layer and for what minimum thickness of the cover layer will light of wavelength $\lambda = 520\,\mathrm{nm}$ pass through the plate without loss. Assume that the light is incident perpendicularly on the wafer and the losses due to absorption of light are negligible.*

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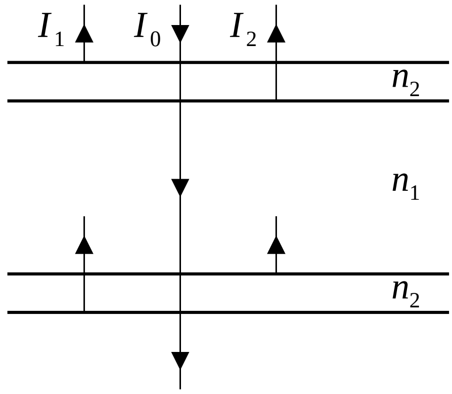


Fig. 2

The light rays that are reflected from the cover layer and the glass plate are coherent because they are produced by splitting a single ray, and the cover layer is thinner than the coherent length of light. Light will pass through without loss if the intensity of the reflected light is zero, that is

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\left(\frac{2\pi}{\lambda}\Delta\right) = 0\ ,$$

where the intensity of light reflected at the interface between the air and the cover layer is

$$I_1 = \left(\frac{n_2 - 1}{n_2 + 1}\right)^2 I_0\ ,$$

and the intensity of light reflected at the interface between the cover layer and the glass plate is

$$I_2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 (I_0 - I_1) \approx \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 I_0\ .$$

The intensity of the reflected light will be zero if the conditions are simultaneously satisfied

$$I_1 = I_2 \qquad \text{and} \qquad \cos\left(\frac{2\pi}{\lambda}\Delta\right) = -1\ .$$

The first condition implies

$$\left(\frac{n_2 - 1}{n_2 + 1}\right)^2 I_0 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 I_0\ .$$

For the refractive index of the cover layer, the following must therefore hold

$$n_2 = \sqrt{n_1} = \sqrt{1{,}66} = 1{,}30\ .$$

From the second condition for the difference of optical paths must hold

$$\Delta = (2m-1)\frac{\lambda}{2} \ .$$

Since the difference of the optical paths is

$$\Delta = 2hn_2 \ ,$$

for the thickness of the cover layer, it follows

$$h = \frac{(2m-1)\lambda}{4n_1} \qquad m = 1, 2, 3, \dots \ .$$

The smallest thickness of the cover layer will be at $m = 1$, so it is equal to

$$h_{min} = \frac{\lambda}{4n_1} = \frac{520 \cdot 10^{-9}\,\mathrm{m}}{4 \ . \ 1{,}30} = 100\,\mathrm{nm} \ .$$

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5. *When a perpendicularly parallel beam of violet light with wavelength* $\lambda_1 = 420\,\mathrm{nm}$ *is incident on the slit, the centre of the second dark band can be seen on the screen at an angle* $\alpha_1 = 4°53'$ *from the normal to the plane of the slit. At what angle will the centre of the third dark band be seen if we illuminate the slit with green light of wavelength* $\lambda_2 = 550\,\mathrm{nm}$*?*

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If the light bends at the slit, minima are formed on the shade at the points for which

$$d \sin\alpha = k\lambda \qquad k = 1, 2, 3, \dots \ ,$$

therefore, for the second minimum of light with wavelength $\lambda_1$ it is valid

$$d \sin\alpha_1 = 2\lambda_1$$

and for the third minimum of light with wavelength $\lambda_2$ it is valid

$$d \sin\alpha_2 = 3\lambda_2 \ .$$

Dividing the equations by each other produces a new equation

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{2\lambda_1}{3\lambda_2} \ ,$$

from which it is possible to express

$$\sin\alpha_2 = \frac{3\lambda_2}{2\lambda_1} \sin\alpha_1 \ ,$$

after inserting values

$$\sin\alpha_2 = \frac{3 \ . \ 550 \cdot 10^{-9}\,\mathrm{m}}{2 \ . \ 420 \cdot 10^{-9}\,\mathrm{m}} \ . \ \sin 4°53' = 0{,}167 \ ,$$

which corresponds to the angle

$$\alpha_2 = 9°37' \ .$$

6. *Calculate the illuminance of a small circular area of radius $r << R$ located at a distance $R = 2\,\mathrm{m}$ from a point source with luminous intensity $I = 20\,\mathrm{cd}$, if the normal to the area points to the point source.*

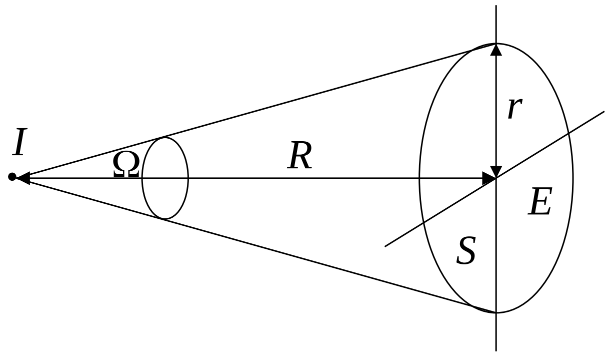


Fig. 3

Illuminance is defined as the ratio of the luminous flux to the size of the area on which the luminous flux falls

$$E = \frac{\Phi}{S} \, .$$

Because a light source whose luminous intensity is $I$ emits a luminous flux to a solid angle $\Omega$

$$\Phi = I\Omega \, ,$$

the illuminance is

$$E = \frac{I\Omega}{S} \, .$$

The condition $r <<< R$ allows to use the relationship between the spherical surface $S$, the radius of the sphere $R$ and the solid angle $\Omega$

$$S = \Omega R^2 \, ,$$

using which the illuminance can be expressed as

$$E = \frac{I\Omega}{\Omega R^2} = \frac{I}{R^2} \, ,$$

after inserting values

$$E = \frac{20\,\mathrm{cd}}{(2\,\mathrm{m})^2} = 5\,\mathrm{lx} \, .$$

7. *A light source whose luminance is* $L = 100\,\mathrm{cd\,m^{-2}}$ *has the shape of a disc with radius* $R = 0{,}5\,\mathrm{m}$. *Calculate the illuminance at a point located at a distance* $a = 5\,\mathrm{m}$ *from the centre of the light source.*

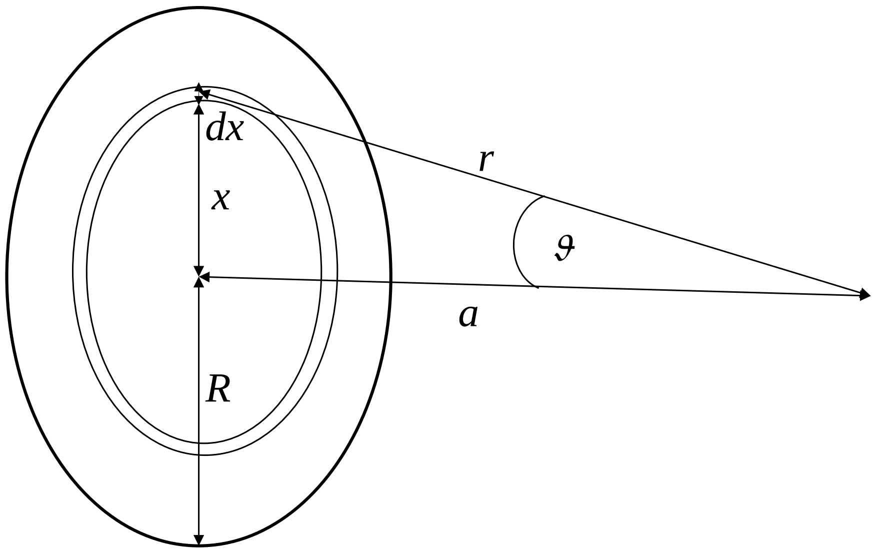


Fig. 4

The light source can be divided into concentric circular rings of radius $x$ and thickness $\mathrm{d}x$, whose area is

$$\mathrm{d}S = 2\pi x \mathrm{d}x\ .$$

The luminous intensity of the ring in the direction to the given point is

$$\mathrm{d}L_\vartheta = L\mathrm{d}S\cos\vartheta = L2\pi x\mathrm{d}x\cos\vartheta$$

and the illuminance produced by the ring at a given point is

$$\mathrm{d}E = \frac{\cos\vartheta}{r^2}L2\pi x\mathrm{d}x\cos\vartheta = \frac{\cos^2\vartheta}{r^2}L2\pi x\mathrm{d}x\ .$$

The distance of the ring from the point is

$$r = \sqrt{x^2 + a^2}\ ,$$

and is also valid

$$\cos\vartheta = \frac{a}{r} = \frac{a}{\sqrt{x^2 + a^2}}\ ,$$

which can be used to express the illuminance produced by the ring as

$$E = 2\pi L a^2 \frac{x}{(x^2 + a^2)^2} \mathrm{d}x$$

and the illuminance produced by the whole light source can be calculated by integrating

$$E = 2\pi L a^2 \int_0^R \frac{x}{(x^2 + a^2)^2} \mathrm{d}x = \pi L a^2 \left[ -\frac{1}{x^2 + a^2} \right]_0^R = \frac{\pi L R^2}{R^2 + a^2} \ ,$$

after inserting values

$$E = \frac{\pi \ . \ 100 \, \mathrm{cd\,m^{-2}} \ . \ (0{,}5 \, \mathrm{m})^2}{(5 \, \mathrm{m})^2 + (0{,}5 \, \mathrm{m})^2} = 3{,}11 \, \mathrm{lx} \ .$$

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8. *A wall is illuminated by two identical bulbs side by side at a distance $d = 2\,\mathrm{m}$ from the wall. When one bulb is switched off, calculate how the second bulb must move to keep the illuminance of the wall the same as before.*

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For the illuminance of an area by a point source whose luminous intensity is $I$, the following holds

$$E = \frac{I}{r^2} \cos \alpha \ .$$

Since the distance of the bulb from the wall is $d$ and the light rays fall perpendicular to the wall $\alpha = 0°$, the illuminance of the wall from a single bulb is

$$E = \frac{I}{d^2} \ .$$

Since both bulbs have the same luminous intensity and are at the same distance from the wall, the illuminance of the wall from the two bulbs is

$$E_2 = 2E = \frac{2I}{d^2} \ .$$

After switching off one bulb, the second bulb must be moved to a distance $x$ from the wall so that the illuminance of the wall from one bulb

$$E_1 = \frac{I}{x^2}$$

remains the same as when illuminated by two bulbs

$$E_1 = E_2 \ .$$

Thus it must be valid

$$\frac{I}{x^2} = 2\frac{I}{d^2} ,$$

which, for the distance of the bulb from the wall, implies

$$x = \frac{d}{\sqrt{2}} ,$$

after inserting

$$x = \frac{2\,\mathrm{m}}{\sqrt{2}} = 1{,}41\,\mathrm{m} .$$

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9. *The table is illuminated by two bulbs which are placed on the ceiling at a distance $d = 1\,\mathrm{m}$ from each other and height $h = 1{,}5\,\mathrm{m}$ above the table. The luminous intensity of each bulb is $I = 100\,\mathrm{cd}$. Calculate the illuminance a) on the table centered between the bulbs b) on the table directly under one of the bulbs.*

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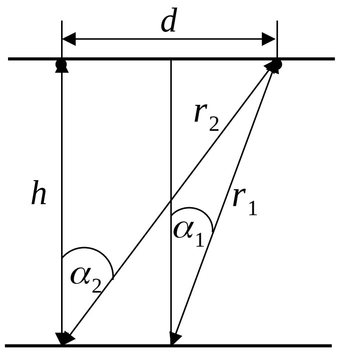


Fig. 5

a) On the table in the middle between the bulbs, the illuminance of the light from each bulb is the same

$$E_1 = E_2 = \frac{I}{r_1^2}\cos\alpha_1 .$$

Because

$$r_1 = \sqrt{h^2 + \left(\frac{d}{2}\right)^2}$$

and also

$$\cos\alpha_1 = \frac{h}{r_1} ,$$

the illuminance from the individual bulbs is

$$E_1 = E_2 = \frac{Ih}{\left[h^2 + \left(\frac{d}{2}\right)^2\right]^{\frac{3}{2}}} \ .$$

The illuminance on the table centered between the bulbs will be the sum of the illuminances from the two bulbs

$$E = E_1 + E_2 = \frac{2Ih}{\left[h^2 + \left(\frac{d}{2}\right)^2\right]^{\frac{3}{2}}} \ ,$$

after inserting values

$$E = \frac{2 \ . \ 100\,\mathrm{cd} \ . \ 1{,}5\,\mathrm{m}}{\left[(1{,}5\,\mathrm{m})^2 + (\frac{1\,\mathrm{m}}{2})^2\right]^{\frac{3}{2}}} = 76\,\mathrm{lx} \ .$$

b) On the table below the bulb, the illuminance will be from the bulb that is directly above it

$$E_1 = \frac{I}{h^2}$$

and the illuminance from the bulb next to it

$$E_2 = \frac{I}{r_2^2} \cos \alpha_2 \ .$$

Because

$$r_2 = \sqrt{d^2 + h^2}$$

and also

$$\cos \alpha_2 = \frac{h}{r_2} \ ,$$

the illuminance from the side bulb can be expressed as

$$E_2 = \frac{Ih}{(d^2 + h^2)^{\frac{3}{2}}} \ .$$

The illuminance on the table under one of the bulbs will be the sum of the illuminances from both bulbs

$$E = E_1 + E_2 = \frac{I}{h^2} + \frac{Ih}{(d^2 + h^2)^{\frac{3}{2}}} \ ,$$

after inserting values

$$E = \frac{100\,\mathrm{cd}}{(1{,}5\,\mathrm{m})^2} + \frac{100\,\mathrm{cd} \ . \ 1{,}5\,\mathrm{m}}{[(1\,\mathrm{m})^2 + (1{,}5\,\mathrm{m})^2]^{\frac{3}{2}}} = 70\,\mathrm{lx} \ .$$

10. *In the centre above the circular table top with radius* $R = 80\,\mathrm{cm}$ *is a light source with luminous intensity* $I = 100\,\mathrm{cd}$. *At what height above the table should the light source be placed so that the illuminance of the edge of the table is maximized? What is the maximum illuminance of the edge of the table?*

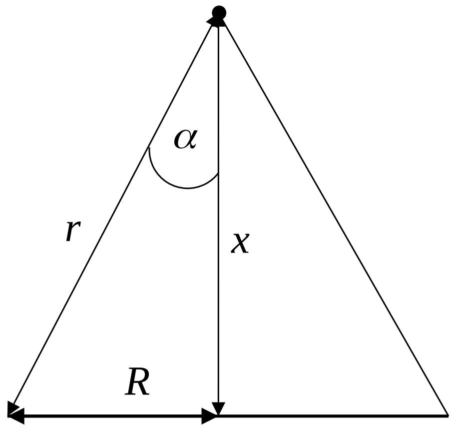


Fig. 6

The illuminance of the table edge is

$$E = \frac{I}{r^2} \cos\alpha \; .$$

Because

$$r^2 = R^2 + x^2 \; ,$$

and also

$$\cos\alpha = \frac{x}{r} = \frac{x}{\sqrt{R^2 + x^2}} \; ,$$

the illuminance of the table edge can be expressed as

$$E = \frac{Ix}{(R^2 + x^2)^{\frac{3}{2}}} \; .$$

The extremum of this function must satisfy the condition

$$\frac{\mathrm{d}E}{\mathrm{d}x} = 0 \; ,$$

which implies

$$I \frac{(R^2 + x^2)^{\frac{3}{2}} - x\frac{3}{2}(R^2 + x^2)^{\frac{1}{2}} 2x}{(R^2 + x^2)^3} = 0 \; .$$

This condition is satisfied for the distance of the light source from the table

$$x = \frac{R}{\sqrt{2}} \, ,$$

after insertion

$$x = \frac{0{,}8\,\mathrm{m}}{\sqrt{2}} = 0{,}57\,\mathrm{m} \, ,$$

at this distance from the light source, the illuminance at the edge of the table is

$$E = \frac{100\,\mathrm{cd}\ .\ 0{,}57\,\mathrm{m}}{\left[(0{,}8\,\mathrm{m})^2 + (0{,}57\,\mathrm{m})^2\right]^{\frac{3}{2}}} = 60\,\mathrm{lx} \, .$$