Problems in magnetic field

1. *An electric current* $I = 2\,\mathrm{A}$ *flows through a long, straight conductor. What is the magnetic induction of this conductor at a distance* $a = 0{,}5\,\mathrm{m}$ *from it?*

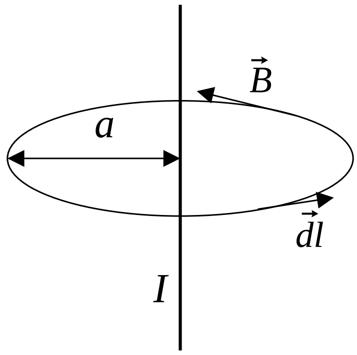

Fig. 1

The solution is possible using Ampère's circuital law, according to which the curve integral of the magnetic induction along any closed oriented curve is equal to the product of the magnetic constant and the electric current that flows through the area bounded by this curve

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 I \ .$$

If a circle with a radius a around the conductor with the current I is chosen as the curve, the vectors $\vec{B}$ and $\mathrm{d}\vec{l}$ will have the same direction and the magnitude of the magnetic induction will be constant on this curve. Therefore, it will apply

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \oint B\mathrm{d}l = B \oint \mathrm{d}l = B2\pi a \ ,$$

from Ampère' s circuical law then follows

$$B2\pi a = \mu_0 I \ ,$$

from which it is possible to express the the magnetic induction as

$$B = \frac{\mu_0 I}{2\pi a} \ .$$

After substituting numerical values

$$B = \frac{4\pi \cdot 10^{-7}\,\mathrm{N\,A^{-2}} \ . \ 2\,\mathrm{A}}{2\pi \ . \ 0{,}5\,\mathrm{m}} = 8 \cdot 10^{-7}\,\mathrm{T} = 0{,}8\,\mu\mathrm{T} \ .$$

2. *An electric current $I = 1{,}5\,\mathrm{A}$ flows through a circular conductor with radius $R = 10\,\mathrm{cm}$. Calculate the magnetic induction of the conductor on its axis at a distance $x = 20\,\mathrm{cm}$ from the center of the circle and at the center of the circle.*

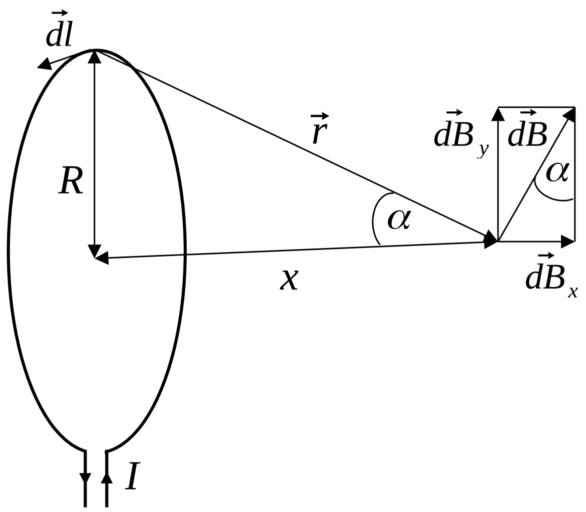

Fig. 2

The calculation of the magnetic induction of the conductor is possible using the Biot-Savart-Laplace law, which allows you to calculate the contribution

$$\mathrm{d}\vec{B} = \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}\vec{l} \times \vec{r}}{r^3} ,$$

to the magnetic induction from an element $\mathrm{d}\vec{l}$ at a location with position $\vec{r}$. Since the vectors $\mathrm{d}\vec{l}$ and $\vec{r}$ are perpendicular, the magnitude of this contribution can be expressed as

$$\mathrm{d}B = \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}l r \sin 90^\circ}{r^3} = \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}l}{r^2} .$$

The vector $\mathrm{d}\vec{B}$ can be decomposed into components

$$\mathrm{d}\vec{B} = \mathrm{d}\vec{B}_x + \mathrm{d}\vec{B}_y .$$

The $\mathrm{d}\vec{B}_y$ components from the $\mathrm{d}\vec{l}$ elements, which lie on the opposite sides of the circle, are the same size and oppositely oriented, therefore they will cancel each other and the resulting magnetic field will be given only by the sum of the $\mathrm{d}\vec{B}_x$ components, the magnitude of which is

$$\mathrm{d}B_x = \mathrm{d}B \sin\alpha = \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}l}{r^2} \sin\alpha .$$

The magnitude of the position vector can be expressed from the Pythagorean theorem

$$r = \sqrt{R^2 + x^2}\ ,$$

in a right triangle also applies

$$\sin\alpha = \frac{R}{r} = \frac{R}{\sqrt{R^2 + x^2}}\ ,$$

from which it follows

$$\mathrm{d}B_x = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}l}{R^2 + x^2}\frac{R}{\sqrt{R^2 + x^2}} = \frac{\mu_0 I}{4\pi}\frac{R}{(R^2 + x^2)^{\frac{3}{2}}}\mathrm{d}l\ .$$

The magnetic induction of the entire conductor can be calculated by integrating over the entire length of the conductor

$$B = \int \mathrm{d}B_x = \int_0^{2\pi R} \frac{\mu_0 I}{4\pi}\frac{R}{(R^2 + x^2)^{\frac{3}{2}}}\mathrm{d}l = \frac{\mu_0 I}{4\pi}\frac{R}{(R^2 + x^2)^{\frac{3}{2}}}\int_0^{2\pi R}\mathrm{d}l =$$

$$= \frac{\mu_0 I}{4\pi}\frac{R}{(R^2 + x^2)^{\frac{3}{2}}}2\pi R = \frac{\mu_0 I R^2}{2(R^2 + x^2)^{\frac{3}{2}}}\ ,$$

after substitution

$$B = \frac{4\pi \cdot 10^{-7}\,\mathrm{N\,A^{-2}}\ .\ 2\,\mathrm{A}\ .\ (0{,}1\,\mathrm{m})^2}{2\left[(0{,}1\,\mathrm{m})^2 + (0{,}2\,\mathrm{m})^2\right]^{\frac{3}{2}}} = 1{,}12 \cdot 10^{-6}\,\mathrm{T} = 1{,}12\,\mu\mathrm{T}\ .$$

The magnetic induction in the center of the circle can be expressed from the resulting relation by substituting $x = 0$, which results in

$$B = \frac{\mu_0 I}{2R} = \frac{4\pi \cdot 10^{-7}\,\mathrm{N\,A^{-2}}\ .\ 2\,\mathrm{A}}{2\ .\ 0{,}1\,\mathrm{m}} = 1{,}26 \cdot 10^{-5}\,\mathrm{T} = 12{,}6\,\mu\mathrm{T}\ .$$

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3. *An electric current flows through a circular conductor with a radius of* $R = 5\,\mathrm{cm}$*, the magnetic induction in the center of the circle is* $B = 5\,\mathrm{mT}$*. What is its magnetic moment?*

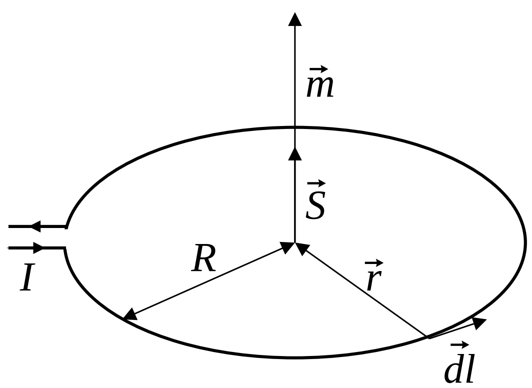

Fig. 3

The magnetic induction of the conductor is given by the Biot-Savart-Laplace law

$$\mathrm{d}\vec{B} = \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}\vec{l} \times \vec{r}}{r^3} ,$$

where an element $\mathrm{d}\vec{l}$ of a conductor with an electric current I at a location with a position vector $\vec{r}$ contributes to the total magnetic induction by the contribution $\mathrm{d}\vec{B}$. Since the vectors $\mathrm{d}\vec{l}$ and $\vec{r}$ are perpendicular, it follows

$$\mathrm{d}B = \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}l r \sin 90°}{r^3} = \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}l}{r^2} .$$

Since the size of the position vector is equal to the radius of the circle, it will be

$$\mathrm{d}B = \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}l}{R^2} .$$

The magnetic induction of the entire conductor can be calculated by integrating over the entire length of the conductor

$$B = \int \mathrm{d}B = \int\limits_0^{2\pi R} \frac{\mu_0 I}{4\pi R^2} \mathrm{d}l = \frac{\mu_0 I}{4\pi R^2} \int\limits_0^{2\pi R} \mathrm{d}l = \frac{\mu_0 I}{4\pi R^2} 2\pi R = \frac{\mu_0 I}{2R} ,$$

which can be used to express the electric current in a conductor

$$I = \frac{2BR}{\mu_0} .$$

The magnitude of the magnetic moment of the planar loop with surface S through which the electric current I flows is

$$m = IS = \frac{2BR}{\mu_0} \pi R^2 = \frac{2\pi B R^3}{\mu_0} ,$$

after substituting numerical values

$$m = \frac{2\pi \,.\, 5 \cdot 10^{-3}\,\mathrm{T} \,.\, (0{,}05\,\mathrm{m})^3}{4\pi \cdot 10^{-7}\,\mathrm{N\,A^{-2}}} = 3{,}125\,\mathrm{A\,m^2} .$$

4. *Two long, straight conductors, parallel to each other, carry equal electric currents. The distance between them is* $a = 0{,}5\,\mathrm{m}$*, and the force exerted by one conductor per unit length of the other conductor is* $F_0 = 2\cdot 10^{-7}\,\mathrm{N\,m^{-1}}$*. Calculate the electric currents in the conductors.*

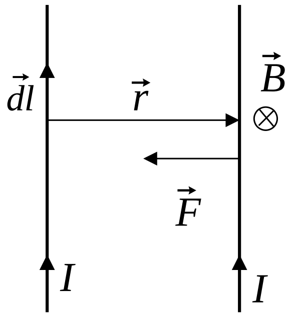

Fig. 4

The magnetic induction created by a straight conductor with an electric current I can be calculated using Ampère's circuital law

$$\oint \vec{B}\cdot \mathrm{d}\vec{l} = \mu_0 I \ ,$$

if the integration loop is a circle with a radius a with one conductor in the centre, it follows

$$B 2\pi a = \mu_0 I \ ,$$

from which it is possible to express the magnetic induction:

$$B = \frac{\mu_0 I}{2\pi a} \ .$$

The force acting in the magnetic field $\vec{B}$ on the conductor element $\mathrm{d}\vec{l}$ through which the electric current I flows is expressed by Ampère's force law

$$\mathrm{d}\vec{F} = I\mathrm{d}\vec{l} \times \vec{B} \ ,$$

because the conductor element $\mathrm{d}\vec{l}$ and the magnetic induction $\vec{B}$ are perpendicular to each other, the magnitude of the force is

$$\mathrm{d}F = I\mathrm{d}lB \ ,$$

from which it is possible to express the magnitude of the force acting per unit length of the conductor

$$F_0 = \frac{\mathrm{d}F}{\mathrm{d}l} = IB \; .$$

After inserting the magnetic induction created by the second conductor

$$F_0 = \frac{\mu_0 I^2}{2\pi a} \; ,$$

from which it is possible to calculate the electric current

$$I = \sqrt{\frac{2\pi a F_0}{\mu_0}} \; ,$$

after inserting numerical values

$$I = \sqrt{\frac{2\pi \; . \; 0{,}5\,\mathrm{m} \; . \; 2\cdot 10^{-7}\,\mathrm{N\,m^{-1}}}{4\pi\cdot 10^{-7}\,\mathrm{N\,A^{-2}}}} = 0{,}707\,\mathrm{A} \; .$$

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5. *Countercurrent electric currents* $I_1 = I_2 = 5\,\mathrm{A}$ *flow through two coaxial copper tubes in vacuum with radii* $R_1 = 5\,\mathrm{mm}$ *and* $R_2 = 10\,\mathrm{mm}$. *What is the magnetic induction at distances* $r_2 = 3\,\mathrm{mm}$, $r_2 = 8\,\mathrm{mm}$ *and* $r_3 = 15\,\mathrm{mm}$ *from of the common axis of the tubes?*

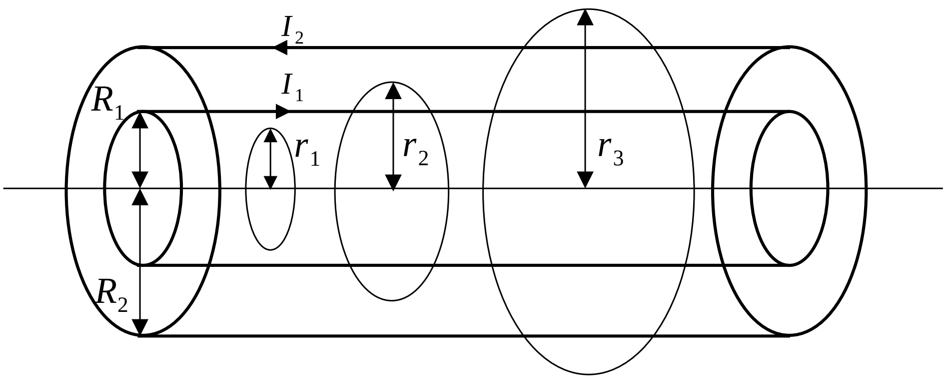

Fig. 5

The magnetic induction can be calculated using Ampère's circuital law

$$\oint \vec{B}\cdot \mathrm{d}\vec{l} = \mu_0 I_{\mathrm{net}} \; .$$

If the integration curve is a circle with radius r_1, zero electric current flows through this curve, which implies

$$B_1 \oint_0^{2\pi r_1} \mathrm{d}l = 0 \; ,$$

$$B_1 2\pi r_1 = 0\ .$$

Therefore the magnetic induction at the distance r_1 will be

$$B_1 = 0\,\mathrm{T}\ .$$

If the integration loop is a circle with radius r_2, the electric current I_1 flows through this curve, which implies

$$B_2 \oint_0^{2\pi r_2} \mathrm{d}l = \mu_0 I_1\ ,$$

$$B_2 2\pi r_2 = \mu_0 I_1\ .$$

Therefore the magnetic induction at the distance r_2 will be

$$B_2 = \frac{\mu_0 I_1}{2\pi r_2} = \frac{4\pi \cdot 10^{-7}\,\mathrm{N\,A^{-2}}\ .\ 5\,\mathrm{A}}{2\pi\ .\ 0{,}008\,\mathrm{m}} = 1{,}25 \cdot 10^{-3}\,\mathrm{T} = 0{,}125\,\mathrm{mT}\ .$$

If the integration loop is a circle with radius r_3, the net electric current flows through this curve is $I_1 - I_2 = 0$, which implies

$$B_3 \oint_0^{2\pi r_3} \mathrm{d}l = \mu_0 (I_1 - I_2)\ ,$$

$$B_3 2\pi r_3 = 0\ .$$

Therefore the magnetic induction at the distance r_3 will be

$$B_3 = 0\,\mathrm{T}\ .$$

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6. *In a homogeneous magnetic field with magnetic induction $B = 0{,}5\,\mathrm{T}$, there is a rectangular conductor with sides $a = 5\,\mathrm{cm}$ and $b = 3\,\mathrm{cm}$, through which flows an electric current $I = 1\,\mathrm{A}$. The conductor can rotate around an axis that passes through the centers of the sides b and is perpendicular to the magnetic induction. What work will be done by the external forces that turn the conductor by an angle $\alpha = 90°$ from the stable position?*

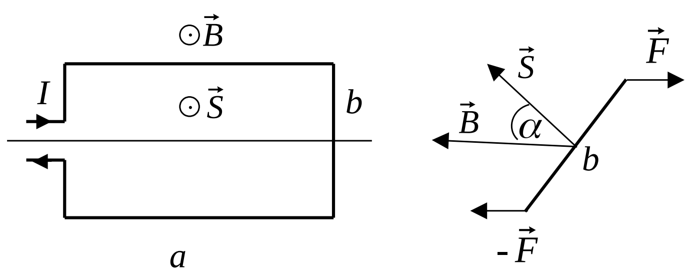

Fig. 6

A magnetic field exerts on a closed loop with electric current a torque

$$\vec{M} = \vec{m} \times \vec{B} \ ,$$

where the magnetic moment of the loop is

$$\vec{m} = I\vec{S} \ .$$

The following applies

$$\vec{M} = I\vec{S} \times \vec{B} \ .$$

The magnetic field tries to rotate the loop so that the area vector of the loop and the magnetic induction vector have the same direction, this position is stable. In order for the loop to rotate about its axis, it must be acted upon by a couple of external forces that do the work when the loop rotates by $\mathrm{d}\alpha$

$$\mathrm{d}A = M\mathrm{d}\alpha \ .$$

The torque of the couple of external forces must be equal to the torque of the magnetic field

$$M = ISB\sin\alpha \ ,$$

therefore, the work of the couple of forces when turning the loop by $\mathrm{d}\alpha$ will be

$$\mathrm{d}A = ISB\sin\alpha\mathrm{d}\alpha$$

and the total work during the rotation by the angle α will be

$$A = \int\limits_0^\alpha ISB\sin\alpha\mathrm{d}\alpha = ISB\int\limits_0^\alpha \sin\alpha\mathrm{d}\alpha = ISB\left[-\cos\alpha\right]_0^\alpha = ISB(1-\cos\alpha) \ .$$

Because the area of the loop is

$$S = ab \ ,$$

it is possible to express the work as

$$A = IabB(1-\cos\alpha) \ ,$$

after substitution

$$A = 1\,\mathrm{A}\ .\ 0{,}05\,\mathrm{m}\ .\ 0{,}03\,\mathrm{m}\ .\ 0{,}5\,\mathrm{T}\ .\ (1-\cos 90^\circ) = 7{,}5\cdot 10^{-4}\,\mathrm{J} = 0{,}75\,\mathrm{mJ} \ .$$

7. *The toroid coil with a radius* $R = 10\,\mathrm{cm}$ *and the cross section radius* $r = 1\,\mathrm{cm}$ *has* $N = 10\,000$ *turns wound on a steel core through which an electric current* $I = 1\,\mathrm{A}$ *flows. The magnetic flux through the cross section of the core is* $\Phi = 7{,}5\,\mathrm{mWb}$. *Calculate the relative permeability of the core.*

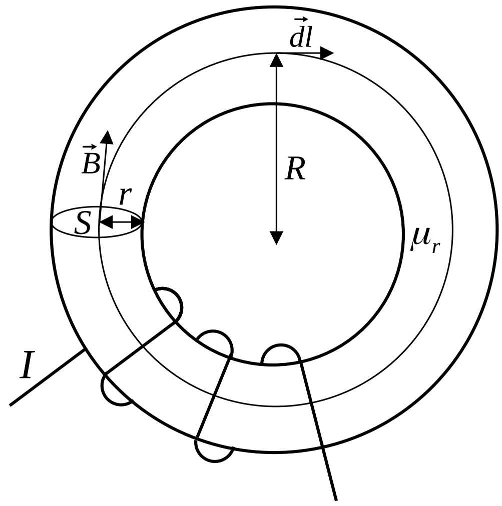

Fig. 7

The magnetic induction can be calculated using Ampère's circuital law

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_r \mu_0 I_{\text{net}} \ .$$

If the integration curve is a circle with radius R centered at the center of the toroid, the net electric current flowing through this curve is

$$I_{\text{net}} = NI \ .$$

The vectors $\vec{B}$ and $\mathrm{d}\vec{l}$ have the same direction and the magnitude of the magnetic induction is constant, therefore

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \oint_0^{2\pi R} B \mathrm{d}l = B \oint_0^{2\pi R} \mathrm{d}l = 2\pi R B \ ,$$

then from Ampèrovho circuital law follows

$$2\pi R B = \mu_r \mu_0 N I \ ,$$

from which the relative permeability can be expressed as

$$\mu_r = \frac{2\pi R B}{\mu_0 N I} \ .$$

From the magnetic induction flux

$$\Phi = BS \ ,$$

for the magnetic induction follows

$$B = \frac{\Phi}{S} = \frac{\Phi}{\pi r^2} \ ,$$

using which the relative permeability can be calculated as

$$\mu_r = \frac{2\pi R\Phi}{\mu_0 N I \pi r^2} = \frac{2R\Phi}{\mu_0 N I r^2} \ ,$$

after substitution

$$\mu_r = \frac{2 \ . \ 0{,}1\,\mathrm{m} \ . \ 7{,}5 \cdot 10^{-3}\,\mathrm{Wb}}{4\pi \cdot 10^{-7}\,\mathrm{N\,A^{-2}} \ . \ 10000 \ . \ 1\,\mathrm{A} \ . \ (0{,}01\,\mathrm{m})^2} = 1194 \ .$$

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8. *What is the energy of the magnetic field of a toroid with radius* $R = 20\,\mathrm{cm}$ *on which* $N = 5000$ *turns with radius* $r = 1\,\mathrm{cm}$ *are wound, when an electric current* $I = 5\,\mathrm{mA}$ *flows through it?*

The magnetic induction can be calculated using Ampère's circuital law

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 I_{\mathrm{net}} \ .$$

If the integration curve is a circle with radius R centered at the center of the toroid, the net electric current flowing through this curve is

$$I_{\mathrm{net}} = NI \ ,$$

The vectors $\vec{B}$ and $\mathrm{d}\vec{l}$ have the same direction and the magnitude of the magnetic induction is constant, therefore

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \oint_0^{2\pi R} B\mathrm{d}l = B \oint_0^{2\pi R} \mathrm{d}l = B2\pi R \ ,$$

then from Ampèrovho circuital law follows

$$B2\pi R = \mu_0 NI \ ,$$

from which the magnetic induction can be expressed as

$$B = \frac{\mu_0 NI}{2\pi R} \ .$$

The magnetic flux through one turn is

$$\Phi = \int\limits_S \vec{B} \cdot \mathrm{d}\vec{S} = B \int\limits_S \mathrm{d}S = BS = B\pi r^2$$

and the magnetic flux through N turns of the toroid is

$$\Phi_{\text{total}} = N\Phi = NB\pi r^2 \ ,$$

After substituting the magnetic induction, the magnetic flux is

$$\Phi_{\text{total}} = \frac{\mu_0 N^2 I r^2}{2R} \ ,$$

by which the inductance of the toroid can be calculated as

$$L = \frac{\Phi_{\text{total}}}{I} = \frac{\mu_0 N^2 r^2}{2R} \ .$$

The energy of the magnetic field in the toroid is

$$W_m = \frac{1}{2} L I^2 = \frac{\mu_0 N^2 r^2 I^2}{4R} \ ,$$

after substitution

$$W_m = \frac{4\pi \cdot 10^{-7}\,\mathrm{N\,A^{-2}} \ . \ (5000)^2 \ . \ (0{,}01\,\mathrm{m})^2 \ . \ (0{,}005\,\mathrm{A})^2}{4 \ . \ 0{,}2\,\mathrm{m}} =$$

$$= 9{,}81 \cdot 10^{-8}\,\mathrm{J} = 98{,}1\,\mathrm{nJ} \ .$$

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9. *A rectangular conductor with sides $a = 20\,\mathrm{cm}$ and $b = 10\,\mathrm{cm}$ s located in the Earth's magnetic field with the magnetic induction $B = 45\,\mu\mathrm{T}$. The conductor can rotate about an axis that passes through the center of side b and is perpendicular to the magnetic induction. What is the waveform and the amplitude of the induced voltage in the conductor if the conductor rotates in the magnetic field with a frequency of $f = 50 s^{-1}$.*

According to Faraday's law of induction, the induced electromotive voltage is equal to the negative time change of the magnetic flux

$$U_i = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \ ,$$

where the magnetic flux is defined as

$$\Phi = \int\limits_S \vec{B} \cdot \mathrm{d}\vec{S} = \int\limits_S B \mathrm{d}S \cos\alpha \ .$$

Since the magnetic field induction is constant

$$\Phi = B \int_S \mathrm{d}S \cos\alpha = BS\cos\alpha = Bab\cos\alpha \ ,$$

and the angle between the loop surface vector and magnetic induction vector is

$$\alpha = \omega t = 2\pi f t \ ,$$

thus the magnetic induction flux will be

$$\Phi = Bab\cos(2\pi f t) \ .$$

Faraday's law of electromagnetic induction therefore for the induced voltage follows

$$U_i = -\frac{\mathrm{d}\left[Bab\cos(2\pi f t)\right]}{\mathrm{d}t} = Bab2\pi f\sin(2\pi f t) = U_0\sin(2\pi f t) \ ,$$

where the amplitude of the induced voltage is

$$U_0 = Bab2\pi f \ ,$$

after substituting numerical values

$$U_0 = 45\cdot 10^{-6}\,\mathrm{T}\ .\ 0{,}2\,\mathrm{m}\ .\ 0{,}1\,\mathrm{m}\ .\ 2\pi\ .\ 50\,\mathrm{s}^{-1} = 2{,}83\cdot 10^{-4}\,\mathrm{V} = 0{,}283\,\mathrm{mV} \ .$$

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10. *Calculate the amplitude and the waveform of the induced electric current in a rectangular copper conductor with sides* $a = 10\,\mathrm{cm}$ *and* $b = 4\,\mathrm{cm}$ *cross section* $S = 2\,\mathrm{mm}^2$ *and resistivity* $\rho = 1{,}7\cdot 10^{-8}\,\Omega\,\mathrm{m}$, *which in a homogeneous magnetic field with by induction* $B = 5\,\mathrm{mT}$ *rotates with a frequency* $f = 100\,\mathrm{s}^{-1}$.

According to Faraday's law of induction, the induced electromotive voltage is equal to the negative time change of the magnetic flux through the loop

$$U_i = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \ .$$

where the magnetic flux through the loop is

$$\Phi = \int_S \vec{B}\cdot\mathrm{d}\vec{S} = \int_S B\mathrm{d}S\cos\alpha = B\int_S \mathrm{d}S\cos\alpha = BS\cos\alpha = Bab\cos\alpha \ ,$$

because the angle between the loop surface vector and magnetic induction vector is

$$\alpha = \omega t = 2\pi f t \ ,$$

the magnetic flux will be

$$\Phi = Bab\cos\left(2\pi ft\right) .$$

The induced voltage then follows

$$U_i = -\frac{\mathrm{d}\left[Bab\cos\left(2\pi ft\right)\right]}{\mathrm{d}t} = Bab2\pi f\sin(2\pi ft) .$$

The relationship between induced voltage and current can be expressed using Ohm's law

$$I_i = \frac{U_i}{R} ,$$

where the electrical resistance of the conductor can be calculated as

$$R = \rho\frac{l}{S} = \rho\frac{2(a+b)}{S} .$$

The waveform of the induced electric current will be

$$I_i = \frac{BabS\pi f}{\rho(a+b)}\sin(2\pi ft) = I_0\sin(2\pi ft) ,$$

and its amplitude will be

$$I_i = \frac{BabS\pi f}{\rho(a+b)} ,$$

after substituting numerical values

$$I_0 = \frac{5\cdot 10^{-3}\,\mathrm{T}\ .\ 0{,}1\,\mathrm{m}\ .\ 0{,}04\,\mathrm{m}\ .\ 2\cdot 10^{-6}\,\mathrm{m}^2\ .\ \pi\ .\ 100\,\mathrm{s}^{-1}}{1{,}7\cdot 10^{-8}\,\Omega\,\mathrm{m}\ .\ (0{,}1\,\mathrm{m}+0{,}04\,\mathrm{m})} = 5{,}28\,\mathrm{A} .$$