Mechanics of rigid body

1. *Find the position of the center of gravity of a body formed by cutting a semicircle with radius $b/2$ from a homogeneous rectangle with sides a, b on a side of length b and attaching it to the opposite side of the rectangle.*

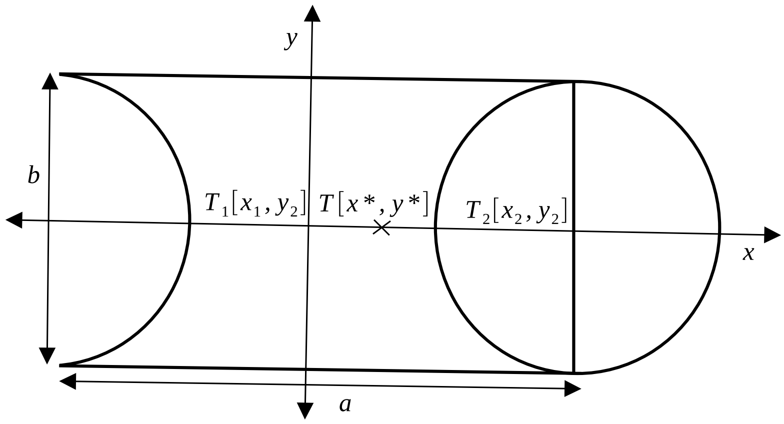

Fig. 1

It is advisable to choose the coordinate system so that its origin is located in the center of the original rectangle, the x axis is parallel to the side a and the y axis to the side b of the rectangle. The position of the center of gravity of the body can be determined as the common center of gravity of two symmetrical bodies, whose centers of gravity are located at their centers of symmetry. The first body will have the shape of a rectangle with semicircular cutouts on the sides with a length of b. The coordinates of the center of gravity of the first body will be

$$x_1 = 0 \ ,$$

$$y_1 = 0 \ .$$

The mass of the first body can be calculated as

$$m_1 = S_1 h\rho = \left[ab - \pi \left(\frac{b}{2}\right)^2\right] h\rho \ ,$$

where h denotes the thickness of the body and ρ its density. The second body will be a circle with radius $b/2$. The coordinates of the center of mass of the second body are

$$x_2 = \frac{a}{2} \ ,$$

$$y_2 = 0\ .$$

The mass of the second body can be calculated as

$$m_2 = S_2 h\rho = \pi\left(\frac{b}{2}\right)^2 h\rho\ .$$

For the position of the common center of gravity of two bodies

$$\vec{r^*} = \frac{\sum\limits_i m_i\vec{r_i}}{\sum\limits_i m_i} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2}\ ,$$

therefore, the coordinates of the common center of gravity of these two bodies will be

$$x^* = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{\pi\left(\frac{b}{2}\right)^2 h\rho\frac{a}{2}}{\left[ab - \pi\left(\frac{b}{2}\right)^2\right]h\rho + \pi\left(\frac{b}{2}\right)^2 h\rho} = \frac{\pi b}{8}\ ,$$

$$y^* = 0\ .$$

---

2. *Find the position of the center of gravity of a homogeneous hemisphere with radius* $R = 10\,\mathrm{cm}$.

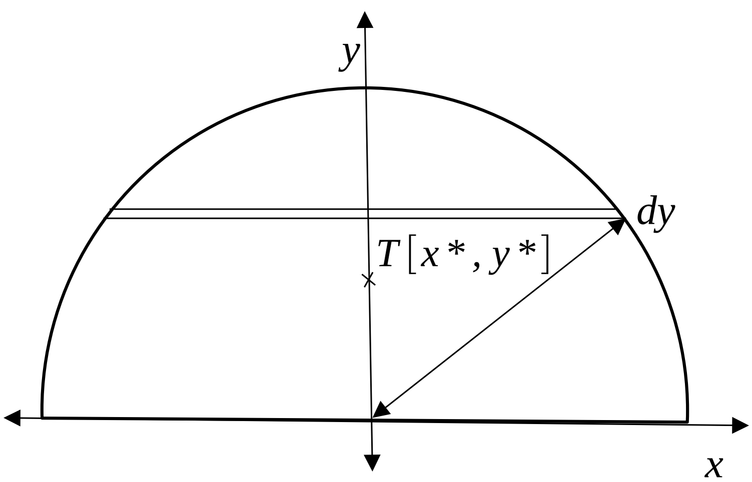

Fig. 2

It is advisable to choose the coordinate system so that its origin is located in the center of the circular base of the hemisphere and the x and z axes lie in its plane. The y axis is the axis of symmetry of the body, therefore the center of gravity of the body will be located on this axis and it will be valid for it

$$x^* = 0\ ,$$

$$z^* = 0 \; .$$

The position of the center of gravity of the body is

$$\vec{r^*} = \frac{1}{M} \int\limits_M \vec{r}\, dm \; ,$$

where M is the mass of the body. For the coordinate of the center of gravity in the direction of the y axis, therefore

$$y^* = \frac{1}{M} \int\limits_M y\, dm = \frac{1}{V\rho} \int\limits_V y\rho\, dV = \frac{1}{V} \int\limits_V y\, dV \; ,$$

where ρ is the density of the hemisphere and the volume of the hemisphere is

$$V = \frac{2}{3}\pi R^3 \; .$$

The volume element of the hemisphere has the shape of a circular disc with radius x and thickness $\mathrm{d}y$

$$dV = \pi x^2 dy = \pi(R^2 - y^2) dy \; ,$$

where the Pythagorean theorem was used

$$R^2 = x^2 + y^2 \; .$$

The position of the center of mass in the direction of the y axis will therefore be

$$y^* = \frac{3}{2\pi R^3} \int\limits_0^R y\pi(R^2 - y^2) dy = \frac{3}{2\pi R^3} \left[\pi R^2 \frac{y^2}{2} - \pi \frac{y^4}{4} \right]_0^R =$$

$$= \frac{3}{2\pi R^3} \left(\pi \frac{R^4}{2} - \pi \frac{R^4}{4} \right) = \frac{3}{8} R \; .$$

After substituting numerical values

$$y^* = \frac{3}{8}\, 10\,\mathrm{cm} = 3{,}78\,\mathrm{cm} \; .$$

3. *A father and a son carry a load on a rod of length $l = 2\,\mathrm{m}$. How far from the father's end of the rod should the load be hung so that the father carries three times as much force as the son? Compared to the mass of the load, the mass of the rod is negligible.*

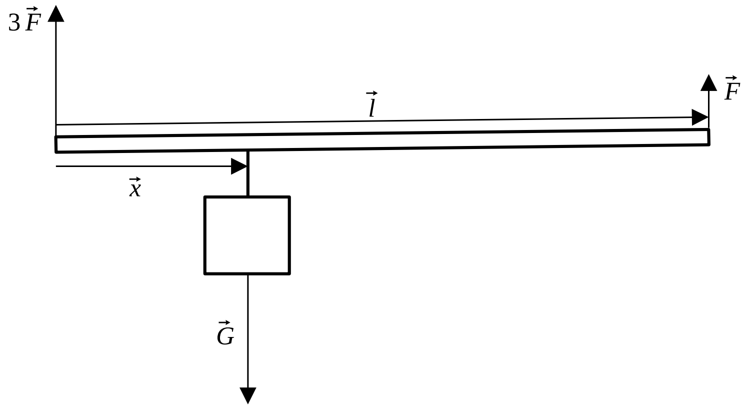

Fig. 3

A body is in equilibrium if the vector sum of all external forces acting on the body is equal to zero

$$\sum_i \vec{F}_i = 0$$

and simultaneously the vector sum of the moments of all external forces with respect to an arbitrary point is equal to zero

$$\sum_i \vec{M}_i = 0\ ,$$

$$\sum_i \vec{r}_i \times \vec{F}_i = 0\ .$$

The external forces acting on the rod (Fig. 11) are the gravity of the load $\vec{G}$, the force of the son $\vec{F}$, and the force of the father $3\vec{F}$. The equilibrium condition for these forces implies

$$\vec{G} + \vec{F} + 3\vec{F} = 0\ .$$

Since the forces of the son and father have opposite directions to the gravitational force, the magnitudes of the forces will be

$$G - 4F = 0\ ,$$

which implies

$$F = \frac{G}{4}\ .$$

The equilibrium condition for the moments of the forces with respect to the point at the father will be

$$\vec{x} \times \vec{G} + \vec{l} \times \vec{F} = 0\ ,$$

the magnitudes of the moments of the forces will be

$$xG \sin 90^\circ + lF \sin(-90^\circ) = 0\ ,$$

$$xG - lF = 0\ .$$

After substituting from the first equilibrium condition

$$xG - l\frac{G}{4} = 0\ ,$$

it is possible to express the distance from the father

$$x = \frac{l}{4}\ .$$

After substituting numerical values

$$x = \frac{2\,\mathrm{m}}{4} = 0{,}5\,\mathrm{m}\ .$$

---

4. *A homogeneous narrow board with length* $l = 5\,\mathrm{m}$ *and mass* $m = 30\,\mathrm{kg}$ *is loaded at one end with a load of mass* $m' = 10\,\mathrm{kg}$. *At what distance from this end should a support be placed so that the plate remains horizontal?*

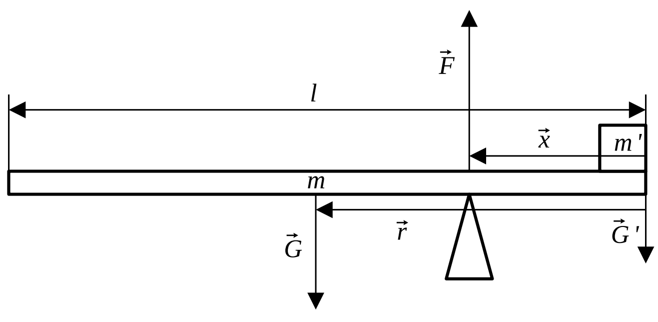

Fig. 4

A body is in equilibrium if the vector sum of all external forces acting on the body is equal to zero

$$\sum_i \vec{F}_i = 0$$

and simultaneously the vector sum of the moments of all external forces with respect to an arbitrary point is equal to zero

$$\sum_i \vec{M}_i = 0 \ ,$$

$$\sum_i \vec{r}_i \times \vec{F}_i = 0 \ ,$$

The external forces acting on the board (Fig. 12) are the gravity of the board $\vec{G}$ and the gravity of the load $\vec{G}'$ and the force of the support F. The equilibrium condition for these forces implies

$$\vec{G} + \vec{G}' + \vec{F} = 0 \ .$$

Since the gravity and the force of the support have opposite directions, the magnitudes of the forces will be

$$G + G' - F = 0 \ ,$$

which implies

$$F = G + G' \ .$$

The equilibrium condition for the moments of the forces with respect to the point at the load will be

$$\vec{r} \times \vec{G} + \vec{x} \times \vec{F} = 0 \ ,$$

for magnitudes of the moments of the forces follow

$$\frac{l}{2} G \sin 90^\circ + xF \sin(-90^\circ) = 0 \ ,$$

$$\frac{l}{2} G - xF = 0 \ .$$

After substituting from the first equilibrium condition

$$\frac{l}{2} G - x(G + G') = 0 \ ,$$

it is possible to express the distance from the load

$$x = \frac{Gl}{2(G+G')} = \frac{mgl}{2(mg+m'g)} = \frac{ml}{2(m+m')} \ ,$$

after substituting numerical values

$$x = \frac{30\,\mathrm{kg}\ .\ 5\,\mathrm{m}}{2(30\,\mathrm{kg}+10\,\mathrm{kg})} = 1{,}875\,\mathrm{m}\ .$$

---

5. *Calculate the moment of inertia of a homogeneous rod of length* $l = 3\,\mathrm{m}$ *and mass* $m = 5\,\mathrm{kg}$ *with respect to an axis passing through the center of gravity of the rod and with respect to an axis passing through the end of the rod.*

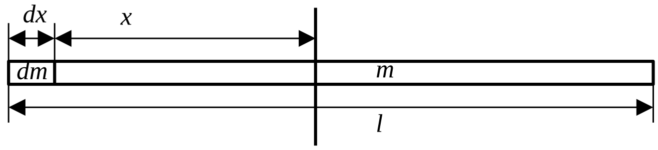

Fig. 5

The moment of inertia for a rigid body can be calculated using the definition

$$J = \int\limits_m r^2 dm\ .$$

If the coordinate system has its origin at the center of the rod and the direction of the x axis is the same as the direction of the rod, the moment of inertia will be

$$J = \int\limits_m x^2 dm\ .$$

Since the rod is homogeneous, its length density is

$$\lambda = \frac{m}{l}\ ,$$

which can be used to express the mass element

$$dm = \lambda dx\ .$$

The moment of inertia of the rod about the axis through the center of gravity will therefore be

$$J^* = \lambda \int\limits_{-\frac{l}{2}}^{+\frac{l}{2}} x^2 dx = \lambda \left[\frac{x^3}{3}\right]_{-\frac{l}{2}}^{+\frac{l}{2}} = \lambda\left(\frac{l^3}{24}+\frac{l^3}{24}\right) = \lambda\frac{l^3}{12}\ .$$

After substituting the length density, the moment of inertia will be

$$J^* = \frac{ml^2}{12}$$

and after substituting the numerical values

$$J^* = \frac{5\,\mathrm{kg}\;.\;(3\,\mathrm{m})^2}{12} = 3{,}75\,\mathrm{kg\,m^2}\;.$$

The moment of inertia about an axis passing through the end of the rod can be calculated using Steiner's theorem

$$J = J^* + ma^2\;,$$

where the distance between the center of gravity and the end of the rod is

$$a = \frac{l}{2}\;.$$

The moment of inertia of the rod about the axis passing through the end of the rod will be

$$J = \frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{3}\;.$$

After substituting numerical values

$$J^* = \frac{5\,\mathrm{kg}\;.\;(3\,\mathrm{m})^2}{3} = 15\,\mathrm{kg\,m^2}\;.$$

---

6. *Calculate the moment of inertia of a homogeneous cylinder of mass* $m = 5\,\mathrm{kg}$ *with radius* $R = 1\,\mathrm{m}$ *with respect to both the axis identical to the axis of symmetry of the cylinder and the parallel axis passing through the edge of the cylinder.*

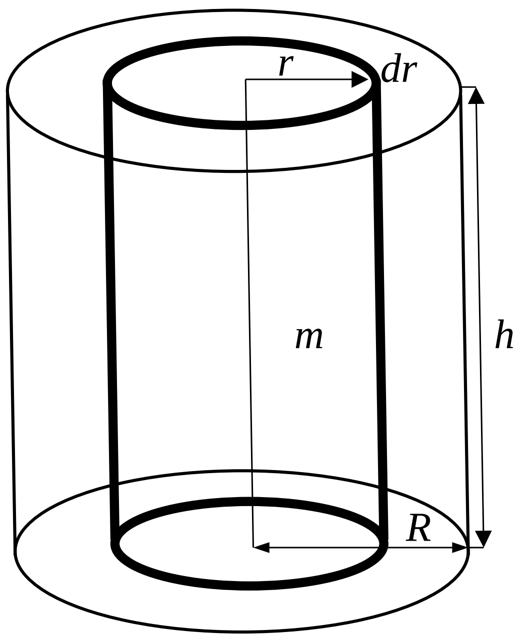

Fig. 6

The moment of inertia of a rigid body can be calculated by the definition

$$J = \int_m r^2 dm \; .$$

If the cylinder is homogeneous, its density can be expressed as

$$\rho = \frac{m}{V} = \frac{m}{\pi R^2 h} \; ,$$

where h is the height of the cylinder. The mass element will be the shell of the cylinder with radius r and thickness $\mathrm{d}r$

$$dm = \rho dV = \rho 2\pi r h dr \; .$$

The moment of inertia of the cylinder with respect to the axis passing through the center of cylinder will be

$$J^* = \rho \int_0^R r^2 2\pi r h \, dr = \rho \left[\pi h \frac{r^4}{2} \right]_0^R = \rho \pi h \frac{R^4}{2} \; ,$$

after substituting the density of the cylinder

$$J^* = \frac{mR^2}{2}$$

and after substituting the numerical values

$$J^* = \frac{5\,\mathrm{kg} \; . \; (1\,\mathrm{m})^2}{2} = 2{,}5\,\mathrm{kg\,m^2} \; .$$

When calculating the moment of inertia with respect to the axis passing through the edge of the cylinder, it is possible to use Steiner's theorem

$$J = J^* + ma^2 \; ,$$

where a is the distance between the axes of rotation passing through the center and the edge of the cylinder

$$a = R \; .$$

The moment of inertia with respect to the axis passing through the edge of the cylinder will be

$$J = \frac{mR^2}{2} + mR^2 = \frac{3mR^2}{2} \; ,$$

and after substituting numerical values

$$J^* = \frac{3\ .\ 5\,\mathrm{kg}\ .\ (1\,\mathrm{m})^2}{2} = 7{,}5\,\mathrm{kg\,m^2}\ .$$

---

7. *A figure skater rotates with a frequency of $f_1 = 3\,\mathrm{s}^{-1}$. At what frequency will the figure skater rotate if he doubles his moment of inertia by extending his arms?*

In an isolated system, the law of conservation of angular momentum states

$$\vec{L}_1 = \vec{L}_2\ ,$$

z ktorého vyplýva

$$J_1\vec{\omega}_1 = J_2\vec{\omega}_2\ ,$$

where J_1, J_2 are the moments of inertia and $\vec{\omega}_1, \vec{\omega}_2$ are the angular velocities of the body before and after the arms are extended. The direction of the angular velocity does not change. From the law of conservation of angular momentum follows

$$J_1\omega_1 = J_2\omega_2\ .$$

The angular velocity can be expressed using the frequency

$$\omega = 2\pi f\ ,$$

then it will be valid

$$J_1 2\pi f_1 = J_2 2\pi f_2\ ,$$

$$J_1 f_1 = J_2 f_2\ .$$

If the moment of inertia is doubled

$$J_2 = 2J_1\ ,$$

the resulting frequency will be halved

$$f_2 = f_1\frac{J_1}{J_2} = \frac{f_1}{2}\ ,$$

after substituting numerical values

$$f_2 = \frac{3\,\mathrm{s}^{-1}}{2} = 1{,}5\,\mathrm{s}^{-1}\ .$$

8. *A gyroscope with a moment of inertia* $J = 10\,\mathrm{kg\,m^2}$ *is rotated from rest by a force, whose moment with respect to the axis of rotation is* $M = 200\,\mathrm{N\,m}$. *In what time will the gyroscope reach a frequency* $f = 8\,\mathrm{s^{-1}}$ *and what will be its kinetic energy then?*

The motion of a gyroscope is described by the equation of motion of a rotating rigid body

$$\vec{M} = J\vec{\alpha}\ ,$$

from which the magnitude of the angular acceleration is

$$\alpha = \frac{M}{J}\ .$$

If the angular acceleration is constant, the angular velocity of the body is

$$\omega = \int \alpha \mathrm{d}t = \alpha t + c\ .$$

Initially the body was at rest, therefore

$$\omega(t = 0\,\mathrm{s}) = 0 \implies c = 0\ ,$$

so the angular velocity will be

$$\omega = \alpha t\ ,$$

which implies for the time of the rotation

$$t = \frac{\omega}{\alpha}\ ,$$

after substituting the angular acceleration, the time of the rotation is

$$t = \omega\frac{J}{M}\ .$$

The relationship between angular velocity and frequency

$$\omega = 2\pi f\ ,$$

allows to modify the relationship for the time of the rotation to the form

$$t = 2\pi f\frac{J}{M}\ ,$$

after substituting numerical values

$$t = 2\,\pi\,.\,8\,\mathrm{s}^{-1}\,.\,\frac{10\,\mathrm{kg\,m}^2}{200\,\mathrm{N\,m}} = 2{,}51\,\mathrm{s}\,.$$

The kinetic energy of the rotating gyroscope will be

$$E_k = \frac{J\omega^2}{2} = \frac{J(2\pi f)^2}{2} = 2J\pi^2 f^2\,,$$

after substituting numerical values

$$E_k = 2\,.\,10\,\mathrm{kg\,m}^2\,.\,\pi^2\,.\,(8\,\mathrm{s}^{-1})^2 = 12\,633\,\mathrm{J}\,.$$

---

9. *Calculate the kinetic energy of a cylindrical body with radius* $R = 10\,\mathrm{cm}$ *and mass* $m = 2\,\mathrm{kg}$ *at time* $t = 10\,\mathrm{s}$. *The body began to rotate from rest around its geometric axis with constant angular acceleration* $\alpha = \pi/8\,\mathrm{s}^{-2}$.

The kinetic energy of a rotating rigid body is

$$E_k = \frac{J\omega^2}{2}\,,$$

where J denotes the moment of inertia with respect to the axis of rotation and ω the angular velocity of the body. The moment of inertia of a homogeneous cylindrical body with respect to its geometric axis (Problem 3.6) can be calculated as

$$J = \frac{mR^2}{2}\,.$$

If the angular acceleration is constant, the angular velocity of the body is

$$\omega = \int \alpha \mathrm{d}t = \alpha t + c\,.$$

Initially the body was at rest, therefore

$$\omega(t = 0\,\mathrm{s}) = 0 \implies c = 0\,,$$

so the angular velocity of the body will be

$$\omega = \alpha t\,.$$

The kinetic energy of the body will be

$$E_k = \frac{1}{2}\frac{mR^2}{2}(at)^2 = \frac{mR^2\alpha^2 t^2}{4}\,,$$

and after substituting numerical values

$$E_k = \frac{2\,\mathrm{kg}\,.\,(0{,}1\,\mathrm{m})^2.\left(\frac{\pi}{8}\mathrm{s}^{-2}\right)^2.(10\,\mathrm{s})^2}{4} = 0{,}077\,\mathrm{J}\,.$$

10. *A rod of length* $l = 1\,\mathrm{m}$ *hangs vertically on an axis passing through its endpoint. What minimum velocity must be given to the free end of the rod to bring it to a horizontal position?*

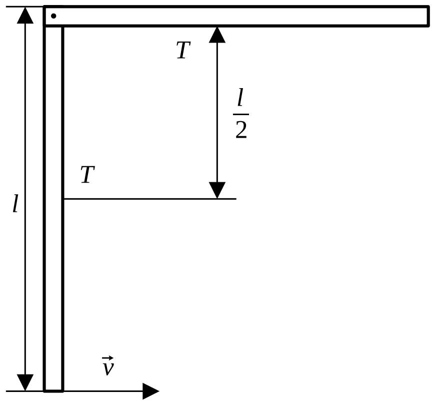

Fig. 7

According to the law of conservation of mechanical energy, the total mechanical energy in an isolated system does not change

$$E_{k1} + E_{p1} = E_{k2} + E_{p2} \ .$$

If the rod in the vertical position has zero potential energy and the kinetic energy of the rod in the horizontal position is zero, the law of conservation of mechanical energy simplifies to the form

$$E_{k1} = E_{p2} \ .$$

The kinetic energy in the vertical position can be calculated as the kinetic energy of a rotating rigid body

$$E_{k1} = \frac{J\omega^2}{2} \ ,$$

where the moment of inertia of the rod with respect to the axis passing through its endpoint (Problem 3.5) will be

$$J = \frac{ml^2}{3}$$

and the angular velocity of the rotating rod can be expressed in terms of the velocity of the rod's endpoint as

$$\omega = \frac{v}{l} \ .$$

In the horizontal position, the potential energy of the rod will be

$$E_{p2} = mgh \ ,$$

where the center of gravity of the rod is raised to a height of

$$h = \frac{l}{2} \ .$$

From the law of conservation of mechanical energy, it follows

$$\frac{1}{2}\frac{ml^2}{3}\left(\frac{v}{l}\right)^2 = mg\frac{l}{2} \ ,$$

from which it is possible to express the velocity of the end point of the rod

$$v = \sqrt{3gl}$$

and after substituting numerical values

$$v = \sqrt{3 \ . \ 9{,}81\,\mathrm{m\,s^{-2}} \ . \ 1\,\mathrm{m}} = 5{,}42\,\mathrm{m\,s^{-1}} \ .$$